# ACOUSTIC OSCILLATIONS IN THE COMBUSTION OF AN AEROSOL

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Thermal acoustic vibrations can be sustained during combustion as a result of an external heat supply, a flow of internal energy, and a flow of kinetic energy [1, 2]. Such vibrations occur in the propagation of flames in gas mixtures and aerosols [4]. In [5], acoustic vibrations were seen in the combustion of an aerosol near the closed end of a firebox. These oscillations were attributed [2, 5] to the feedback mechanism that is the basis for formation of the mixture. Acoustic vibrations were obtained in [6] in the numerical solution of a unidimensional problem on the combustion of an aerosol of a specified composition near the closed end of a tube. In [7], investigators numerically studied a two-dimensional problem on the combustion of an aerosol in a closed volume.

Here, we analytically study the occurrence of vibrations in the combustion of aerosols within bounded volumes. We will use the method of two-scale expansions [8]. In accordance with this method, we introduce a small parameter $\varepsilon$ which is proportional to the mass concentration and calorific value of the fuel. At $\varepsilon \ll 1$, the velocity of the gas is much lower than the speed of sound, and the kinetic energy is negligible. It is shown that oscillations of the parameters occur about values, averaged over rapid time, which satisfy the equations of the homobaric approximation [9, 10]. Here, even with a constant external heat supply ensuring averaged motion, oscillations are generated due to the flow of internal energy from the combustion zone.

1. Basic Equations. Formulation of the Problem. The convective combustion of aerosols is described in the general case by the equations of the mechanics of multiphase media [11]. If we ignore the volume content of particles, consider that the particles are immobile during the initial stage of propagation of the convective front, and assume that their temperature is constant during combustion, then the equations for describing combustion reduce to the equations of gas dynamics with distributed sources of mass and heat [10]. In a rectangular coordinate system, we will examine the region D of a volume V which, for the sake of definiteness, includes the coordinate origin. The volume has the boundary surface $\Sigma$. Let combustion begin from the subregion $D_0$ of the region D. The subregion also includes the origin and has the boundary surface $\Sigma_0$ and volume $V_0$. Similarly to [9, 10], we will examine two variants of the problem: combustion occurs only in the region of initiation $D_0$; hot gases flowing out of $D_0$ form a convective combustion front on which particles are instantaneously ignited. Here, combustion takes place in the region $D_W(t)$, which is of the volume $V_W(t)$ and has the boundary surface $\Sigma_W(t)$. Let $\mathbf{r}_W(t)$ be a radius vector with the pole at the coordinate origin, the end of the vector being located on $\Sigma_W$. Let $\mathbf{r}$ be a radius vector which coincides with $\mathbf{r}_W$ in terms of direction but is of arbitrary length. The latter vector takes values from zero to infinity. At the initial moment of time $\mathbf{r}_W(0) = \mathbf{r}_0$ ($\mathbf{r}_0$ is the radius vector whose end lies on $\Sigma_0$).

We will change over to dimensionless variables. The space variables are referred to the characteristic dimension of the region $\ell$, velocity is referred to the initial sonic velocity in the gas $a_{10}$, and density is referred to the initial densities of the gas and solid phases $\rho_{10}$ and $\rho_{20}$. The dimensionless derived variables are $l/a_{10}$ for time and $\rho_{10}a_{10}^2$ for pressure. The equations for describing the combustion of a unit volume of fuel take the form [10]

$$\frac{\partial\rho_1}{\partial t} + \nabla\rho_1\mathbf{v} = \frac{\varepsilon}{q(\gamma-1)}J, \quad \frac{d\rho_2}{dt} = -\frac{m_{10}\varepsilon}{m_{20}(\gamma-1)q}J,$$
$$\rho_1\frac{\partial\mathbf{v}}{\partial t} + \rho_1(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = -\frac{\varepsilon}{q(\gamma-1)}J\mathbf{v},$$
$$\frac{\partial p}{\partial t} + \gamma\nabla p\mathbf{v} - (\gamma-1)(\mathbf{v}\cdot\nabla)p = \varepsilon J + \frac{\varepsilon^2}{2q}Jv^2,$$

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$$J = \rho_2^{2/3}(\gamma p)^{\psi}\chi(r_w(t) - r), \quad \frac{d\mathbf{r}_w}{dt} = \mathbf{v}_w(\mathbf{r}_w, t)$$
$$\left(\varepsilon = \frac{(\gamma - 1)\, n_0 \pi d_0^2 \rho_2^0 u_s l q}{\rho_{10} a_{10}}, \quad q = \frac{Q}{a_{10}^2}, \quad m_{20} = \frac{\rho_{20}}{\rho_{10} + \rho_{20}}, \quad m_{10} = 1 - m_{20}\right). \tag{1.1}$$

Here, $n_0$ is the number of particles per unit volume; $d_0$, initial diameter of the particles; $\rho_2{}^0$, true density of the solid phase; $u_s$, linear rate of combustion of unit fuel; $\psi$, an empirical constant; $m_{20}$, mass concentration of the solid phase; Q is determined by the enthalpy of the combustion products; $\gamma$, adiabatic exponent; $\varepsilon$ and q, governing dimensionless parameters; $r_W$ and r, moduli of the corresponding vectors; $\chi$, unit function which is equal to unity inside the region $D_W$ and zero outside it; $\mathbf{v}_W$, gas velocity on the surface $\Sigma_W$. In the first variant of the problem, $\mathbf{r}_W \equiv \mathbf{r}_0$. The term J in the right side of the momentum equation accounts for the fact that the gas is sent into the combustion region at zero velocity (at the velocity of the particles at rest).

For example, let the boundary of the region D be completely closed. Then the condition of impermeability prevails on the boundary surface $\Sigma$. Thus, we write the initial and boundary conditions of the problem in the form

$$t = 0\colon \mathbf{v} = 0,\ \rho_1 = 1,\ \rho_2 = 1,\ p = 1/\gamma,\ r_w = r_0;\ t \geqslant 0\colon v_n|_{\Sigma} = 0. \tag{1.2}$$

The velocity vector $\mathbf{v}$ can be represented as the sum of the potential and vortical components. The following relations [12] are valid for each of these components:

$$\mathbf{v} = \nabla\varphi + \operatorname{rot}\mathbf{A}\ (\operatorname{rot}\operatorname{rot}\mathbf{A} = 2\boldsymbol{\omega},\ \boldsymbol{\omega} = 0.5\operatorname{rot}\mathbf{v}),$$
$$\Delta\varphi = \nabla\mathbf{v},\ \Delta\mathbf{A} = -2\boldsymbol{\omega}, \tag{1.3}$$

where is the velocity potential; $\mathbf{A}$ is the vector potential; $\boldsymbol{\omega}$ is the curl.

2. Asymptotic Solution of the Problem at $\varepsilon \ll 1$. We will examine the case of low combustion rates (low fuel concentratons), when $\varepsilon \ll 1$. This is among the class of physical problems in which a small perturbation acts over a long period of time. In these problems, a solution constructed in the form of an ordinary expansion connected with the limiting process $\varepsilon \to 0$ (with t fixed) will not be uniformly valid. The nonuniformity of the expansion will be especially evident when the solution contains secular terms of the type $\varepsilon t$. The physical phenomenon described by the formulation of the present problem is characterized by the presence of two time scales: a small scale associated with the propagation of acoustic disturbances inside the region; a large scale connected with the motion of the gas itself inside the region. In accordance with the concept underlying the method of two-scale perturbations, the uniformly valid expansion being sought should explicitly contain time variables referred to these two time scales.

We introduce the slow $\tau = \varepsilon t$ and fast $t' = \varepsilon^{-1}\zeta(\tau)$ time variables. Here, $\zeta(\tau)$ [$\zeta(0) = 0$] is an unknown function chosen on the basis of the need to have expansions of the sought functions that are uniformly valid. We will seek these expansions in the form

$$\rho_1(\mathbf{r}, t, \varepsilon) = R_{10}(\mathbf{r}, t', \tau) + \varepsilon R_{11}(\mathbf{r}, t', \tau) + \varepsilon^2 R_{12}(\mathbf{r}, t', \tau) + \ldots,$$
$$\rho_2(\mathbf{r}, t, \varepsilon) = R_{20}(\mathbf{r}, t', \tau) + \varepsilon R_{21}(\mathbf{r}, t', \tau) + \varepsilon^2 R_{22}(\mathbf{r}, t', \tau) + \ldots,$$
$$\mathbf{v}(\mathbf{r}, t, \varepsilon) = \varepsilon\mathbf{v}_1(\mathbf{r}, t', \tau) + \varepsilon^2\mathbf{v}_2(\mathbf{r}, t', \tau) + \ldots, \tag{2.1}$$
$$p(\mathbf{r}, t, \varepsilon) = P_0(\mathbf{r}, t', \tau) + \varepsilon P_1(\mathbf{r}, t', \tau) + \varepsilon^2 P_2(\mathbf{r}, t', \tau) + \ldots,$$
$$\mathbf{r}_w(t, \varepsilon) = \mathbf{r}_{w0}(t', \tau) + \varepsilon\mathbf{r}_{w1}(t', \tau) + \varepsilon^2\mathbf{r}_{w2}(t', \tau) + \ldots$$

The requirement of uniform validity followed in determining the terms of expansions (2.1) amounts to the condition that the ratio of each given term of the expansion to the preceding term be finite throughout the entire domain of the independent variables being studied.

With the introduction of new independent variables, the operator for differentiation with respect to time takes the form

$$\partial/\partial t = \dot{\zeta}\partial/\partial t' + \varepsilon\partial/\partial\tau \quad (\dot{\zeta} = d\zeta/d\tau). \tag{2.2}$$

Inserting (2.2) into (1.1) and allowing for (2.1) we obtain the following for the zeroth approximation

$$R_{10} = R_{10}(\mathbf{r}, \tau),\ R_{20} = R_{20}(\tau),\ P_0 = P_0(\tau),\ \mathbf{r}_{w0} = \mathbf{r}_{w0}(\tau)$$
$$(R_{10}(\mathbf{r}, 0) = 1,\ R_{20}(0) = 1,\ P_0(0) = 1/\gamma,\ \mathbf{r}_{w0}(0) = \mathbf{r}_0). \tag{2.3}$$

The parentheses contain the initial conditions for the zeroth approximation found on the basis of (1.2).

In accordance with (2.3), in the zeroth approximation the density and the position of the convective front will be independent of the fast time variable, while the pressure will be independent of the space variables (homobaric condition). Pressure will be a function only of the slow time variable. To determine the functions in (2.3), we examine the below equations of the first approximation. These equations follow from (1.1), (2.1), and (2.2):

$$\dot{\zeta}\frac{\partial R_{11}}{\partial t'} = \frac{R_{20}^{2/3}(\gamma P_0)^{\psi}}{q(\gamma-1)}\chi(r_{w0}-r) - \frac{\partial R_{10}}{\partial \tau} - \nabla R_{10}\mathbf{v}_1,$$
$$\dot{\zeta}\frac{\partial R_{21}}{\partial \tau} = -\frac{m_{10}R_{20}^{2/3}(\gamma P_0)^{\psi}}{m_{20}q(\gamma-1)}\chi(r_{w0}-r) - \frac{\partial R_{20}}{\partial \tau},$$
$$\dot{\zeta}R_{10}\frac{\partial \mathbf{v}_1}{\partial t'} + \nabla P_1 = 0, \quad \dot{\mathbf{r}}_{w0} + \dot{\zeta}\frac{\partial \mathbf{r}_{w1}}{\partial t'} = \mathbf{v}_{w1}(\mathbf{r}_{w0}), \tag{2.4}$$
$$\dot{\zeta}\frac{\partial P_1}{\partial t'} + \gamma P_0\nabla\mathbf{v}_1 = R_{20}^{2/3}(\gamma P_0)^{\psi}\chi(r_{w0}-r) - \dot{P}_0$$
$$(R_{11}(\mathbf{r}, 0, 0) = R_{21}(\mathbf{r}, 0, 0) = \mathbf{v}_1(\mathbf{r}, 0, 0) = P_1(\mathbf{r}, 0, 0) = \mathbf{r}_{w1}(0, 0) = 0).$$

The parentheses show the initial conditions for the first approximation. The left sides of the momentum and energy equations in (2.4) represent the acoustic operators of the equations of motion of the gas with variable density $R_{10}$. After separation of the variables, these equations yield

$$\nabla\left(\frac{1}{R_{10}}\nabla f_p\right) + \lambda^2 f_p = 0, \quad \left(\frac{1}{R_{10}}\nabla f_p\right)_n\Big|_{\Sigma} = 0. \tag{2.5}$$

Here, the second relation follows from boundary condition (1.2); $f_p(\mathbf{r}, \tau)$ is a function which is independent of the fast time; $\lambda$ is a nonnegative parameter.

Problem (2.5) is an eigenvalue problem. The eigenfunctions $f_{pi}$ and $f_{pj}$ corresponding to the eigenvalues $\lambda_i$ and $\lambda_j$ form an orthogonal and normalized system [13]

$$\int_D f_{pi}f_{pj}dD = \begin{cases}0, & i \neq j,\\ 1, & i = j,\end{cases} \quad \int_D f_{pi}dD = 0. \tag{2.6}$$

The minimum eigenvalue $\lambda_0 = 0$ corresponds to $f_{p0} = 1$.

Subjecting the momentum equation in (2.4) to the operation of divergence and rotation and using (1.3), we obtain

$$\dot{\zeta}\frac{\partial}{\partial t'}\Delta\varphi_1 + \nabla\frac{1}{R_{10}}\nabla P_1 = 0, \quad \dot{\zeta}\frac{\partial \omega_1}{\partial t'} + \frac{1}{2}\operatorname{rot}\left(\frac{1}{R_{10}}\nabla P_1\right) = 0. \tag{2.7}$$

We expand the unit function, as well as $P_1$ and $\Delta\varphi_1$, into series in eigenfunctions $f_{pk}$:

$$P_1 = P_{10}(t', \tau) + \sum_{k=1}^{\infty} P_{1k}(t', \tau)f_{pk},$$
$$\Delta\varphi_1 = \sum_{k=1}^{\infty}\lambda_k\Phi_{1k}(t', \tau)f_{pk}, \quad \chi(r_{w0}-r) = V_{w0}(\tau) + \sum_{k=1}^{\infty}\chi_k f_{pk} \tag{2.8}$$

($V_{w0}$ is the volume of the combustion zone in the zeroth approximation). Inserting (2.8) into the third equation of (2.4), we have

$$\dot{\zeta}\frac{\partial P_{10}}{\partial t'} + \dot{\zeta}\sum_{k=1}^{\infty}\frac{\partial P_{1k}}{\partial t'}f_{pk} + \gamma P_0\sum_{k=1}^{\infty}\lambda_k\Phi_{1k}f_{pk} = R_{20}^{2/3}(\gamma P_0)^{\psi}\left(V_{w0} + \sum_{k=1}^{\infty}\chi_k f_{pk}\right) - \dot{P}_0. \tag{2.9}$$

Integrating (2.9) over the region D [with the use of (2.6)] and requiring that $P_{10}$ not have a secular term, we find $\partial P_{10}/\partial t' = 0$ and

$$\dot{P}_0 = R_{20}^{2/3} (\gamma P_0)^{\psi} V_{w0}(\tau). \tag{2.10}$$

Inserting (2.8) into the first equation of (2.7), multiplying the resulting equation and (2.9) by $f_{pk}$, and integrating over D with the use of (2.6), we obtain

$$\dot{\zeta}\frac{\partial \Phi_{1k}}{\partial t'} - \lambda_k P_{1k} = 0, \quad \dot{\zeta}\frac{\partial P_{1k}}{\partial t'} + \gamma P_0 \lambda_k \Phi_{1k} = R_{20}^{2/3} (\gamma P_0)^{\psi} \chi_k. \tag{2.11}$$

With the initial conditions $t' = 0$, $P_{1k} = 0$, $\Phi_{1k} = 0$, the solution of (2.11) has the form

$$\Phi_{1k} = \frac{\chi_k}{\lambda_k \gamma P_0} + \Phi'_{1k}, \quad \Phi'_{1k} = c_{1k}(\tau) \sin \lambda_k \frac{\sqrt{\gamma P_0}}{\dot{\zeta}} t' + c_{2k}(\tau) \cos \lambda_k \frac{\sqrt{\gamma P_0}}{\dot{\zeta}} t',$$

$$P_{1k} = \sqrt{\gamma P_0}\left[c_{1k}(\tau) \cos \lambda_k \frac{\sqrt{\gamma P_0}}{\dot{\zeta}} t' - c_{2k}(\tau) \sin \lambda_k \frac{\sqrt{\gamma P_0}}{\dot{\zeta}} t'\right] \tag{2.12}$$

$$\left(c_{1k}(0) = 0, \quad c_{2k}(0) = -\frac{\chi_k}{\lambda_k}\right).$$

The parentheses show the initial conditions for the arbitrary functions of slow time $c_{1k}$, $c_{2k}$. Inserting (2.12) into the first two equations of (2.8) and using the last relation of (2.8), (2.10), and the initial condition for $P_1$, we find

$$\Delta\varphi_1 = \Delta\varphi_1^* + \Delta\varphi_1', \quad P_1 = \sum_{k=1}^{\infty} P_{1k} f_{pk},$$

$$\Delta\varphi_1^* = \frac{1}{\gamma P_0}\left[R_{20}^{2/3} (\gamma P_0)^{\psi} \chi (r_{w0} - r) - \dot{P}_0\right], \quad \Delta\varphi_1' = \sum_{k=1}^{\infty} \lambda_k \Phi'_{1k} f_{pk}, \tag{2.13}$$

from which it follows that the pressure in the first approximation $P_1$ has only fluctuational terms, while the velocity has both a fluctuational (fast) component $\varphi_1'$, associated with $v_1'$, and a monotonic (slow) component $\varphi_1^*$, associated with $v_1^*$ and dependent only on the slow time $\tau$.

Inserting (2.13) into the second equation of (2.7) and integrating over t', we obtain the following for the curl:

$$\omega_1 = \omega_1'(\mathbf{r}, t', \tau) + \omega_1^*(\mathbf{r}, \tau), \quad \omega_1' = -\frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \Phi'_{1k} \operatorname{rot}\left(\frac{1}{R_{10}} \nabla f_{pk}\right) \tag{2.14}$$

($\omega_1^*$ is an arbitrary vector which is independent of the fast time). Since the conditions $\omega_1 = 0$, $R_{10} = 1$ and $\omega_1' = 0$ are satisfied at $t' = \tau = 0$, we take $\omega_1^*(r, 0) = 0$ as the initial condition for $\omega_1^*$.

The fast velocity component is more conveniently determined directly by integrating the momentum equation (2.4) over t'. The resulting arbitrary function $\mathbf{r}$, $\tau$ is obviously the slow velocity component. As a consequence of this,

$$\mathbf{v}_1' = -\sum_{k=1}^{\infty} \frac{\Phi'_{1k}}{\lambda_k R_{10}} \nabla f_{pk}, \quad \mathbf{v}_1^* = \nabla\varphi_1^* + \operatorname{rot} \mathbf{A}_1^*, \tag{2.15}$$

from which it follows that the fast and slow velocity components are in the general case vortical in character. Inserting (2.15) into the first, second, and fourth equations of (2.4), integrating them over t', and requiring that $R_{11}$ and $R_{21}$ not have any secular terms, we find

$$\frac{\partial R_{10}}{\partial \tau} + \nabla R_{10} \mathbf{v}_1^* = \frac{R_{20}^{2/3} (\gamma P_0)^{\psi}}{q(\gamma - 1)} \chi (r_{w0} - r), \quad \dot{\mathbf{r}}_{w0} = \mathbf{v}_{w1}^*, \quad \frac{\partial R_{20}}{\partial \tau} =$$

$$= -\frac{m_{10} R_{20}^{2/3} (\gamma P_0)^{\psi}}{m_{20} q (\gamma - 1)} \chi (r_{w0} - r), \tag{2.16}$$

$$R_{11} = -\frac{1}{\gamma P_0}\sum_{k=1}^{\infty}\frac{P_{1k}}{\lambda_k^2}\Delta f_{pk} + R_{11}^*(\mathbf{r},\tau) \qquad (R_{11}^*(\mathbf{r},0)=0),\ R_{21} = R_{21}^*(\mathbf{r},\tau),\ R_{21}^*(\mathbf{r},0)=0, \tag{2.16}$$
$$\mathbf{r}_{w1} = \frac{1}{\sqrt{\gamma P_0}}\sum_{k=1}^{\infty}\frac{P_{1k}}{\lambda_k^2 R_{10}}\nabla f_{pk}(\mathbf{r}_{w0}) + \mathbf{r}_{w1}^*(\mathbf{r},\tau) \quad (\mathbf{r}_{w1}^*(\mathbf{r},0)=0).$$

Here, $R_1^*$ and $\mathbf{r}_{w1}^*$ are arbitrary functions independent of fast time; the parentheses show their initial conditions. It follows from (2.16) that the position of the convective front in the zeroth approximation depends only on the slow velocity component. Oscillations of the front's position take place in the first approximation.

Integrating the third relation of (2.13) over $D_w$, using the second relation of (2.16), and using the formula for differentiation, with respect to time, of the integral taken over the moving volume [9], we obtain

$$\frac{dV_{w0}}{d\tau} = \frac{1}{\gamma P_0}\left(R_{20}^{2/3}(\gamma P_0)^{\psi}V_{w0} - \dot{P}_0 V_{w0}\right). \tag{2.17}$$

In the zeroth approximations, Eqs. (2.10) and (2.17) give the law of increase in uniform (over space) pressure and the change in the volume of the combustion zone in the second variant of the problem. In the first variant, the pressure increase in the volume is completely determined by Eq. (2.10). However, a relation analogous to (2.17) is satisfied for the variable volume of the reaction products leaving $D_0$.

In the unidimensional case, when, in accordance with (2.15), the slow velocity $\mathbf{v}_1^*$ is found completely from the third equation of (2.13), Eqs. (2.16) for $R_{10}$ and $\mathbf{r}_{w0}$ are closed and with the equations for pressure $P_0$ give the complete homobaric approximation [9, 10]. We introduce parameters averaged over the fast time, enclosed in brackets. Then $\langle P_1\rangle = 0$, $\langle R_{11}^*\rangle = R_{11}^*$, $\langle \mathbf{v}_1\rangle = \mathbf{v}_1^*$, $\langle \boldsymbol{\omega}_1\rangle = \boldsymbol{\omega}_1^*$, $\langle \mathbf{r}_{w1}\rangle = \mathbf{r}_{w1}^*$. Thus, the functions denoted by an asterisk are flow parameters averaged over the fast time.

Due to the presence of the vortical component, the velocity $\mathbf{v}_1^*$ was not found in the three- and two-dimensional cases. Thus, the parameters of the zeroth approximation $R_{10}$ and $_{w0}$ were also not determined. On the whole, in the first approximation the arbitrary functions $\zeta(\tau)$, $R_{11}^*(\mathbf{r},\tau)$, $R_{21}^*(\mathbf{r},\tau)$, $\mathbf{A}_1^*(\mathbf{r},\tau)$, $\mathbf{r}_{w1}^*(\mathbf{r},\tau)$, $c_{1k}(\tau)$, $c_{2k}(\tau)$ are unknown. To find them, we examine the equations below of the second approximation, which follow from (1.1 (2.1), and (2.2)

$$\dot{\zeta}\frac{\partial R_{21}}{\partial t'} + \nabla R_{10}\mathbf{v}_2 = F_1(\mathbf{r},t',\tau),\quad \dot{\zeta}\frac{\partial R_{22}}{\partial t'} = F_2(\mathbf{r},t',\tau),$$
$$\dot{\zeta}R_{10}\frac{\partial \mathbf{v}_2}{\partial t'} + \nabla P_2 = \mathbf{G}(\mathbf{r},t',\tau),\quad \frac{\partial \mathbf{r}_{w1}}{\partial \tau} + \dot{\zeta}\frac{\partial \mathbf{r}_{w1}}{\partial t'} = \mathbf{v}_{w2},$$
$$\dot{\zeta}\frac{\partial P_2}{\partial t'} + \gamma P_0\nabla\mathbf{v}_2 = H(\mathbf{r},t',\tau),$$
$$F_1 = \frac{R_{20}^{2/3}(\gamma P_0)^{\psi}\delta(r_{w0}-r)}{q(\gamma-1)} + \frac{\chi(r_{w0}-r)}{q(\gamma-1)}\left(\frac{2R_{21}}{3R_{20}} + \frac{\psi P_1}{P_0}\right) + \frac{\partial R_{11}}{\partial\tau} - \nabla R_{11}\mathbf{v}_1, \tag{2.18}$$
$$F_2 = -\frac{m_{10}R_{20}^{2/3}(\gamma P_0)^{\psi}r_{w1}\delta(r_{w0}-r)}{m_{20}q(\gamma-1)} - \frac{m_{10}\chi(r_{w0}-r)}{m_{20}q(\gamma-1)}\left(\frac{2R_{21}}{R_{20}} + \frac{\psi P_1}{P_0}\right),$$
$$\mathbf{G} = -\left(\frac{\partial\mathbf{v}_1}{\partial\tau} + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1\right)R_{10} - \dot{\zeta}R_{11}\frac{\partial\mathbf{v}_1}{\partial t'} - \frac{R_{20}^{2/3}(\gamma P_0)^{\psi}\chi(r_{w0}-r)}{q(\gamma-1)}\mathbf{v}_1,$$
$$H = R_{20}^{2/3}(\gamma P_0)^{\psi}r_{w1}\delta(r_{w0}-r) + \chi(r_{w0}-r)\left(\frac{2R_{21}}{R_{20}} + \frac{\psi P_1}{P_0}\right) - \frac{\partial P_1}{\partial\tau} - \gamma P_1\nabla\mathbf{v}_1 - \mathbf{v}_1\nabla P_1$$

($\delta$ is the delta function). Having transformed the momentum equation in (2.18) to the Gromek–Lamb form, subjecting it to the operation of divergence (here using the formula $\nabla(\mathbf{v}_1 \times 2\boldsymbol{\omega}_1) = 4\boldsymbol{\omega}_1^2 - 2\mathbf{v}_1\,\mathrm{rot}\,\boldsymbol{\omega}_1$ [14] from vector calculus), subjecting the initial form of the equation to rotation, and using (1.3), we obtain

$$\dot{\zeta}\frac{\partial\Delta\varphi_2}{\partial t'}+\nabla\frac{1}{R_{10}}\nabla P_2=\nabla\frac{1}{R_{10}}\mathbf{G},\ \dot{\zeta}\frac{\partial\boldsymbol{\omega}_2}{\partial t'}+\frac{1}{2}\operatorname{rot}\left(\frac{1}{R_{10}}\nabla P_2\right)=\operatorname{rot}\frac{\mathbf{G}}{R_{10}},$$

$$\nabla\frac{\mathbf{G}}{R_{10}}=-\frac{\partial\nabla\mathbf{v}_1}{\partial\tau}-\Delta\frac{\mathbf{v}_1^2}{2}+4\boldsymbol{\omega}_1^2-2\mathbf{v}_1\cdot\operatorname{rot}\boldsymbol{\omega}_1-$$

$$-\dot{\zeta}\nabla\frac{R_1}{R_{10}}\nabla\frac{\partial\mathbf{v}_1}{\partial t'}-\frac{R_{20}^{2/3}(\gamma P_0)^{\psi}}{q(\gamma-1)}\nabla\frac{\mathbf{v}_1\chi(r_{w0}-r)}{R_{10}},\tag{2.19}$$

$$\operatorname{rot}\frac{\mathbf{G}}{R_{10}}=-2\left(\frac{\partial\boldsymbol{\omega}_1}{\partial\tau}+(\mathbf{v}_1\cdot\nabla)\boldsymbol{\omega}_1-(\boldsymbol{\omega}_1\cdot\nabla)\mathbf{v}_1+\boldsymbol{\omega}_1\nabla\mathbf{v}_1\right)-$$

$$-\operatorname{rot}\left(\dot{\zeta}\frac{R_{11}}{R_{10}}\frac{\partial\mathbf{v}_1}{\partial t'}\right)-\frac{R_{20}^{2/3}(\gamma P_0)^{\psi}}{q(\gamma-1)}\operatorname{rot}\frac{\mathbf{v}_1\chi(r_{w0}-r)}{R_{10}}.$$

As in the first approximation, the left sides of the momentum and energy equations in (2.18) are acoustic operators. Thus, $P_2$ and $\Delta\varphi_2$ should be sought in the form of series in $f_{pk}$ which are analogous to (2.8). Using the same procedure employed in deriving Eqs. (2.9) and (2.11), we find the following for the second approximation:

$$\dot{\zeta}\frac{\partial P_{20}}{\partial t'}=\int_D H\,dD,\ \ \dot{\zeta}\lambda_k\frac{\partial\Phi_{2k}}{\partial t'}-\lambda_k^2P_{2k}=\int_D\nabla\left(\frac{1}{R_{10}}\mathbf{G}\right)f_{pk}\,dD,$$

$$\frac{\partial^2\Phi_{2k}}{\partial t'^2}+\lambda_k^2\frac{\gamma P_0}{\dot{\zeta}^2}\Phi_{2k}=\frac{\lambda_k}{\dot{\zeta}^2}\int_D Hf_{pk}\,dD+\frac{1}{\dot{\zeta}\lambda_k}\frac{\partial}{\partial t'}\int_D\nabla\left(\frac{1}{R_{10}}\mathbf{G}\right)f_{pk}\,dD\tag{2.20}$$

[$P_{20}(t', \tau)$, $P_{2k}(t', \tau)$, and $\Phi_{2k}(t', \tau)$ are sought functions in expansions of the type (2.11)].

The right side of the second and third equations of (2.20) contain terms $\partial P_1/\partial\tau$, $\partial\mathbf{v}_1/\partial\tau$ which for an arbitrary function $\zeta(\tau)$ give terms of the type $t'\sin\lambda_k t'$, $t'\cos\lambda_k t'$, which lead to an unbounded (resonance) increase in $\Phi_{2k}$. This type of secularity can be eliminated by having chosen the function $\zeta(\tau)$ so that $\dot{\zeta}=\sqrt{\gamma P_0}$.

The right side of the last equation of (2.20) should not contain terms proportional to $\sin\lambda_k t'$, $\cos\lambda_k t'$, which also lead to the resonance growth of $\Phi_{2k}$. Such terms appear mainly due to the presence of the slow component of velocity $\mathbf{v}_1^*$ and due to multiplication of parameters having only fluctuational components under the condition that the equalities $\lambda_i+\lambda_j=\lambda_k$, $\lambda_i-\lambda_j=\lambda_k$ are possible. In accordance with (2.5), at $\tau=0$ and $R_{10}=1$, the eigenvalues are the numbers $\lambda_k=\pi k$, and the equalities indicated above may be satisfied. At $\tau>0$, when $R_{10}\neq\text{const}$, the eigenvalues $\lambda_k$ are generally not multiples of integers. In this case, the possibility of satisfaction of the equalities in question is not obvious. In any case, equating the complete coefficients with $\sin\lambda_k t'$, $\cos\lambda_k t'$ to zero, we can write an infinite chain of first-order ordinary differential equations for $c_{1k}$, $c_{2k}$ ($k=1, 2, \ldots$). Limiting ourselves to a finite number of terms and using the appropriate initial conditions [see (2.12)], in principle we can always obtain a solution to this system. We will assume that this has been done. Given the thus-chosen values of $c_{1k}$ and $c_{2k}$, the functions $\Phi_{2k}$ will have only fluctuational terms. Using this, we find from the second relation of (2.20) that the functions $P_{2k}$ have only fluctuational components and components dependent on $\tau$. Meanwhile, the components dependent on $\tau$ do not lead to uniformity of the expansion. It then follows from the second equation of (2.19) that secular terms arise for the curl $\boldsymbol{\omega}_2$ only due to the presence of the right side of the equation. To eliminate such terms, the following equation must be satisfied for the slow component of the curl

$$\frac{\partial\boldsymbol{\omega}_1^*}{\partial\tau}+(\mathbf{v}_1\cdot\nabla)\boldsymbol{\omega}_1^*-(\boldsymbol{\omega}_1^*\cdot\nabla)\mathbf{v}_1^*+\boldsymbol{\omega}_1^*\nabla\mathbf{v}_1^*=-\frac{R_{20}^{2/3}(\gamma P_0)^{\psi}}{2q(\gamma-1)}\operatorname{rot}\chi(r_{w0}-r)\frac{\mathbf{v}_1^*}{R_{10}}.\tag{2.21}$$

As a result of this procedure, $\boldsymbol{\omega}_2$, $\Phi_{2k}$, and thus $\mathbf{v}_2$ will have only fluctuational components. In order that $\mathbf{r}_{w2}$ [see the fourth equation of (2.18)] not be associated with a secular term, the condition $\mathbf{r}_{w1}^*(\mathbf{r}, \tau)=\mathbf{r}_{w1}^*(\mathbf{r}, 0)=0$ must be satisfied. It then follows from the first two equations of (2.18) that in order for $R_{21}$ and $R_{22}$ to not have secular terms, the following equations must be satisfied:

$$\frac{\partial R_{11}^*}{\partial\tau}+\nabla R_{11}^*\mathbf{v}_1^*=\frac{2R_{21}^*\chi(r_{w0}-r)}{3R_{20}q(\gamma-1)},\ \frac{\partial R_{21}^*}{\partial\tau}=-\frac{2m_{10}R_{21}^*\chi(r_{w0}-r)}{3m_{20}R_{20}q(\gamma-1)}.$$

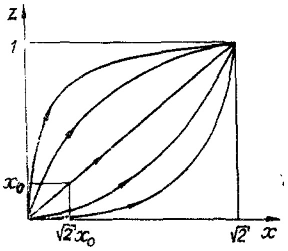


Fig. 1

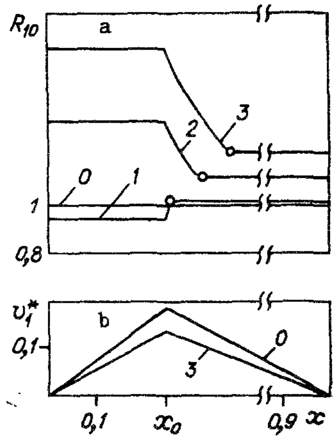


Fig. 2

This completes the selection of the arbitrary (denoted by an asterisk) functions, ensuring uniform validity of the first approximation. Equation (2.21) is an equation of the Helmholtz type. The slow component of the curl $\omega_1^*$ determines the vortical motion averaged over fast time. In accordance with (2.21), averaged vortices develop only behind the combustion front and are due to the difference in velocity between the carrier gas and the gas being supplied for combustion. This difference generates a force during combustion. If this force were not present, i.e., if the combustion products arrived at the velocity of the flow, then the right side of (2.21) would be equal to zero and, in accordance with the Helmholz theorem [12], ${}_1^*$ would be equal to zero at all moments of time. Generally speaking, this is analogous to the effect of friction of the gas against the particles — which, in the presence case of small $\varepsilon$, can be ignored.

It should be noted that, in conformity with (2.14), the fluctuational components of the curl are nontrivial throughout the region of flow behind the convective gas-combustion front. This front does not coincide with the combustion front in the first variant of the problem. Ahead of the convective front — where adiabatic compression occurs — the density of the gas is uniform and flow is nonvortical. In accordance with (2.10) and (2.17), the adiabatic integral is satisfied in this region.

The value of $\omega_1^*$ is determined by the dimensionless parameter $\nu = 1/(2q(\gamma - 1))$, which is small in combustion problems ($\nu \ll 1$). In accordance with (2.21), in the zeroth approximation with respect to this parameter, $\omega_1^*(\mathbf{r}, \tau) = 0$ ($\mathbf{v}_1^* = \nabla\varphi_1^*$). In the first approximation with respect to $\nu$, $\omega_1^*$ depends on a source term in the Helmholtz equation of the form $\nabla\chi(r_{W0} - r) \times \nabla\varphi_1^*$. Thus, at $\nu \ll 1$ ($\varepsilon \ll 1$), the slow component of the curl is negligibly small and the averaged motion is potential flow.

As an example of potential flow, we will examine a problem concerning the motion of a gas in a closed cube $0 \le x, y, z \le 1$. Let gas evolution occur in part of this cube: $0 \le x, y, z \le x_0$, $x_0 < 1$. We adopt the model kinetics of constant gas-evolution rate ($J = 1$), which is asymptotically valid during the initial stage of combustion [10]. The law of pressure change in the cube has the form $P_0 = 1/\gamma + x_0{}^3\tau$. For this region, the solution of the Poisson equation [13] [third equation of (2.13)] is as follows:

$$\varphi_1^* = -\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{lmn}}{\pi^2(l^2+m^2+n^2)}\cos\pi lx\cos\pi my\cos\pi nz,$$
$$a_{000}=0,\ a_{0mn}=\frac{x_0\sin\pi mx_0\sin\pi nx_0}{\gamma P_0\pi^2 mn},\quad a_{l0n}=\frac{x_0\sin\pi lx_0\sin\pi nx_0}{\gamma P_0\pi^2 ln}, \tag{2.22}$$
$$a_{lm0}=\frac{x_0\sin\pi lx_0\sin\pi mx_0}{\gamma P_0\pi^2 lm},\quad a_{lmn}=\frac{\sin\pi lx_0\sin\pi mx_0\sin\pi nx_0}{\pi^3 lmn}.$$

It follows from (2.22) that the direction of the streamlines is independent of time and that all of the streamlines begin at the corner (0, 0, 0) and end at the opposite corner (1, 1, 1). Figure 1 shows a sketch of the streamlines in the plane of the diagonal section of the cube x = y. The solution of the analogous problem in the plane case was given in [7].

Let us examine the change in the amplitude of oscillation of parameters characterizing the fast components of the first approximation, i.e., the relations $c_{1k}(\tau)$, $c_{2k}(\tau)$. For simplicity, we will examine a unidimensional problem. Let combustion occur at a constant rate in the part $0 \le x \le x_0$ of the closed region $0 \le x \le 1$ ($x_0 < 1$). The solution of this problem in the zeroth (homobaric) approximation, i.e., the solution of Eq. (2.10), the third equation of (2.13), and the first equation of (2.16) are written in the form

$$R_{10} = f(\tau)(\gamma P_0)^{1/\gamma},\ x \leqslant x_0;\ R_{10} = f(\xi)(\gamma P_0)^{1/\gamma},\ x \geqslant x_0,\ \xi \geqslant 0;$$
$$R_{10} = (\gamma P_0)^{1/\gamma},\ x \geqslant x_0,\ \xi \leqslant 0;\ P_0(\tau) = 1/\gamma + x_0\tau,$$
$$f(\tau) = (\gamma P_0)^{-1/\gamma x_0} + \frac{(\gamma P_0)^{1-1/\gamma} - (\gamma P_0)^{-1/\gamma x_0}}{q(\gamma-1)(1-x_0+\gamma x_0)},\quad \xi = \frac{1}{\gamma x_0}\left[\gamma P_0\left(\frac{1-x}{1-x_0}\right)^{\gamma} - 1\right], \tag{2.23}$$
$$v_1^* = \frac{1-x_0}{\gamma P_0}x,\ x \leqslant x_0;\ v_1^* = \frac{x_0}{\gamma P_0}(1-x),\quad x \geqslant x_0.$$

Figure 2a, b shows the distributions of density $R_{10}(x)$ and velocity $v_1^*(x)$ of the gas with $q = 20$, $\gamma = 1.4$, $x_0 = 0.25$ at different moments of time ($\tau = 0$, 0.1, 0.5, 1 – lines 0-3). The velocity distribution is linear. The density of the gas in the zone of adiabatic compression ahead of the convective front (with the position of the front denoted by circles) always increases. The density in the combustion zone at the initial moments of time decreases. Then, beginning at a certain moment, it increases and eventually becomes greater than the density in the zone of adiabatic compression. It should be noted that with limitingly large q ($q \to \infty$), $R_{10}$ in the combustion zone decreases for any $\tau$. With finite q, an initial reduction in $R_{10}$ occurs only upon satisfaction of the condition $q(\gamma - 1) > 1/(1 - x_0)$. Otherwise, $R_{10}$ increases from the very beginning.

In analyzing the first approximation, we will restrict ourselves to the first terms of series (2.13), i.e., we will approximate the solution by means of the first eigenfunction $f_{p1}$. Then using the above-described procedure to exclude terms proportional to $\sin\lambda_1 t'$ and $\cos\lambda_1 t'$ from the right side of the third equation of (2.20), we obtain ordinary differential equations

$$\dot{c}_{11} + A_1(\tau)c_{11} = 0,\quad \dot{c}_{21} + A_1(\tau)c_{21} = 0\ \left(c_{11}(0) = 0,\ c_{21}(0) = -\frac{\chi_1}{\lambda_1}\right), \tag{2.24}$$
$$A_1 = \left(\frac{\gamma}{2}+1\right)\int_0^1 \frac{\partial v_1^*}{\partial x} f_{p1}^2 dx + \int_0^1 v_1^* \frac{\partial f_{p1}}{\partial x} f_{p1} dx + \frac{\gamma}{4}\frac{\dot{P}_0}{\gamma P_0} - \frac{1}{2\lambda_1^2}\int_0^1 \frac{\partial^2 v_1^*}{\partial x^2}\frac{\partial f_{p1}}{\partial x} f_{p1} dx +$$
$$+ \frac{1}{2\lambda_1^2 q(\gamma-1)}\int_0^1 \frac{1}{R_{10}}\left(\frac{\partial f_{p1}}{\partial x}\right)^2 dx\quad \left(\chi_1 = \int_0^{x_0} f_{p1} dx\right),$$

from which it follows that $c_{11}(\tau) = 0$ for any $\tau$. Using the last equation of (2.23) and integrating by parts for the integrals in (2.24), we obtain

$$\int_0^1 v_1^* \frac{\partial f_{p1}}{\partial x} f_{p1} dx = -\frac{1}{2}\int_0^1 \frac{\partial v_1^*}{\partial x} f_{p1}^2 dx,\quad \int_0^1 \frac{\partial v_1^*}{\partial x} f_{p1}^2 dx = \frac{1}{\gamma P_0}\left(\int_0^{x_0} f_{p1}^2 dx - x_0\right),$$
$$\int_0^1 \frac{\partial^2 v_1^*}{\partial x^2}\frac{\partial f_{p1}}{\partial x} f_{p1} dx = -\frac{f_{p1}(x_0)}{\gamma P_0}\frac{\partial f_{p1}(x_0)}{\partial x}, \tag{2.25}$$
$$\int_0^{x_0} \frac{1}{R_{10}}\left(\frac{\partial f_{p1}}{\partial x}\right)^2 dx = \left[\frac{f_{p1}}{R_{10}}\frac{\partial f_{p1}}{\partial x}\right]_0^{x_0} + \lambda_1^2\int_0^{x_0} f_{p1}^2 dx.$$

At $\tau = 0$, $R_{10} = 1$ and, in accordance with (2.5), $\lambda_1 = \pi$, $f_{p1} = \cos\pi x$. Then, using (2.25), we find

$$A_1(0) = \frac{\gamma+1}{2}\left(\frac{\sin 2\pi x_0}{4\pi} - \frac{x_0}{2}\right) - \frac{\sin 2\pi x_0}{4\pi} + \frac{\gamma x_0}{4} + \frac{1}{2q(\gamma-1)}\left(\frac{x_0}{2} - \frac{\sin 2\pi x_0}{4\pi}\right).$$

It is evident from the last relation that at $\gamma < 2$, the function $A_1(0) < 0$ for any $x_0 < 1$. At large $\tau$ ($\tau \to \infty$), we have $P_0$, $R_{10} \to \infty$, and all of the terms of $A_1$ except the last tend to zero. Meanwhile, at large $\tau$, $A_1$ is finite and positive. Thus, in accordance with Eq. (2.24), the amplitude of oscillations of the parameters $c_{21}$ increases due to heat and gas liberation. The amplitude of the oscillations decreases after a certain period of time has elapsed. The oscillations decay due to the action of the force that develops from the difference in the velocities of the carrier gas and the gas given off during combustion. Here, the reduction in amplitude occurs as $\exp(-\tau)$, which agrees with data from numerical solution of the problem in [6].

Equations of the type (2.24) can be obtained for the analogous two- and three-dimensional problems. In the general case, when the averaged flow is vortical, it is difficult to analyze the change in the amplitude of the vibrations due to the lack of an analytical solution to Eqs. (2.21) for $\omega_1{}^*$. At $\nu \ll 1$, when flow is potential in the zeroth approximation with respect to $\nu$, it follows from equations analogous to (2.24) that the amplitude of the oscillations reaches a steady-state value after increasing during the initial stage. Such behavior of the amplitude is connected with the fact that the force which leads to decay of the oscillations is on the order of $\nu$, and $\nu$ is automatically zero in the zeroth approximation. Decay of the oscillations should be manifest in the first approximation with respect to $\nu$. The study of this approximation involves the solution of Eq. (2.21). It should be noted that in [7], where the analogous two-dimensional problem with small $\varepsilon$ and $\nu$ was solved numerically, it was shown that the parameters of the flow undergo decaying oscillations about the solution obtained in the homobaric approximation.

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